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UNIVERSITÉ DE MONTRÉAL
ÉCOLE DES HAUTES ÉTUDES COMMERCIALES
ÉCOLE POLYTECHNIQUE

CENTRE DE RECHERCHE SUR LES TRANSPORTS
UNIVERSITÉ DE MONTRÉAL
C.P. 6128, SUCCURSALE A
MONTRÉAL
CANADA H3C 3J7

TÉLÉPHONE (514) 343-7575
TÉLEX 05-24146 BIBPOLYTEC
EMAIL crt@crt.umontreal.ca
TÉLÉCOPIEUR (FAX) (514) 343-7121

90-06428/

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# THE WINDY POSTMAN PROBLEM ON GENERAL GRAPHS

by

Michel Gendreau[1]
Gilbert Laporte[1]
and
Yongchang Zhao[1, 2]

[1] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, succursale A, Montréal, Québec, Canada H3C 3J7.

[2] The Systems Engineering Research Institute, Shanghai Institute of Mechanical Engineering, Shanghai, People's Republic of China.

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## ABSTRACT

The Windy Postman Problem consists of determining a least cost oriented tour covering all edges of a connected undirected graph, when the cost of traversing an edge depends on the direction of travel. New integer linear programming formulations for this problem are suggested, and alternative solution procedures derived from them are presented.



Key words : Windy postman problem, Eulerian graphs.


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## RÉSUMÉ

Le Problème du Postier Chinois Orienté consiste à traverser à moindre coût toutes les arrêtes d'un graphe connexe et non orienté, lorsque le coût de passage sur une arrête dépend de la direction du parcours. On présente dans cet article de nouvelles formulations de programmation linéaire en nombres entiers pour ce problème, ainsi que des méthodes de solutions résultant de ces formulations.



Mots-clés : Problème du postier chinois orienté, graphes eulériens.

## 1. Introduction

The Windy Postman Problem (WPP) can be defined as follows. Let G = (V,E) be an undirected connected graph with two costs $c_{ij}$ and $c_{ji}$ associated with each edge ij of E. In order to simplify the presentation, assume G has no parallel edges. This assumption has no significant impact on our results. The digraph (directed graph) $D_G$= (V,A) is derived from G by defining A as the set of all arcs (i,j) and (j,i) such that ij$\in$E. Assume that $D_G$ contains no cycle of negative cost. A path is a sequence p = ($i_0$ $i_1$, $i_1$ $i_2$, ..., $i_{k-1}i_k$) of edges of E. A dipath is a sequence q = (($i_0$, $i_1$), ($i_1$, $i_2$), ..., ($i_{k-1}$, $i_k$)) of arcs of A. A (di)path is closed if and only if $i_0 = i_k$. A diorientation of a path p is a dipath q which substitutes for each edge ij one of the two arcs (i,j) or (j,i). A postman tour of G is a closed path p which contains every edge at least once. A windy postman tour (WPT) is a diorientation of a postman tour. The WPP consists of determining a minimum cost WPT on G.

The WPP corresponds to several real-life situations in which the cost of traversing an edge depends on the direction of travel, as when going with or against the wind, uphill or downhill, etc. This problem was first introduced by Minieka [3]. Guan [2] showed that it is NP-hard and that a minimum cost WPT can be determined in polynomial time for a special class of graphs in which the cost of traversing each cycle C in one direction is equal to the cost of traversing C in the opposite direction. Win [5] concluded that the WPP is solvable in polynomial time if G is an undirected Eulerian graph and he proposed an exact polynomial algorithm for this case. In addition, he developed a heuristic algorithm for non-Eulerian graphs.

The object of this note is to propose new integer linear programming formulations (ILP), as well as alternative solution procedures derived from them and valid for non-Eulerian WPPs. By so doing, we expect to gain new theoretical insight into this difficult problem. Emphasis is placed on properties rather than on numerical results; these will be the object of a forthcoming full-length computational study.

## 2. Integer Programming Formulations

Win [5] proposed the following ILP formulation for the WPP. Let $\delta(i)$ be the set of edges incident to vertex i, and define variables $x_{ij}$ equal to the number of times edge ij is traversed from i to j in the optimal solution. The problem is then to:

(WPP1) Minimize $\sum_{ij \in E} (c_{ij}\, x_{ij} + c_{ji}\, x_{ji})$

subject to

$$x_{ij} + x_{ji} \geq 1 \qquad (ij \in E) \qquad (1)$$

$$\sum_{ij \in \delta(i)} (x_{ij} - x_{ji}) = 0 \qquad (i \in V) \qquad (2)$$

$$x_{ij},\ x_{ji} \geq 0 \qquad (ij \in E) \qquad (3)$$

$$x_{ij},\ x_{ji} \text{ integer} \qquad (ij \in E). \qquad (4)$$

The constraints of this model are self-explanatory. Win proved or stated the following results.

**Proposition 1.** Let P be the polyhedron of the vectors $x = (x_{ij}, x_{ji})$ satisfying (1), (2) and (3). If G is Eulerian, then every vertex of P is integral; otherwise, P has at least one fractional vertex. In general, the components of the vertices of P are either 0, ½, or a positive integer. ∎

**Proposition 2.** Let WP(G) be the convex hull of the incidence vectors of the WPTs of G, i.e. the convex hull of the integral vectors $x = (x_{ij}, x_{ji})$ satisfying (1), (2) and (3). If G is Eulerian, then (1), (2), (3) is a complete description of WP(G). ∎

We now develop new results for (WPP1).

**Proposition 3.** In any optimal (WPP1) solution, exactly one of the following three cases occurs for any edge ij : (i) $x_{ij} = 0$ and $x_{ji} \geq 1$; (ii) $x_{ji} = 0$ and $x_{ij} \geq 1$; (iii) $x_{ij} = x_{ji} = 1$.

**Proof.** The case $x_{ij} = x_{ji} = 0$ is infeasible. In order to show that $x_{ij} = x_{ji} = 1$ whenever $x_{ij} > 0$ and $x_{ji} > 0$, it suffices to observe that if any two variables $x_{ij}$ and $x_{ji}$ were both greater than 1, they could be reduced by 1 without loss of feasibility. Since $D_G$ contains no negative cycles, this cannot occur in an optimal solution. ∎

New valid inequalities can also be derived. For $S \subseteq V$, define $\bar{S} = V \setminus S$, $(S,\bar{S}) = \{(i,j) : i \in S \text{ and } j \in \bar{S}\}$ and let $|(S,S)|$ denote the cardinality of $(S,\bar{S})$. Furthermore, S is said to be <u>odd</u> if it contains an odd number of vertices of odd degree. Note that if S is odd, so is $\bar{S}$, and $|(S,\bar{S})|$ is also an odd number.

**Proposition 4.** The constraints

$$\sum_{(i,j) \in (S,S)} x_{ij} \geq \tfrac{1}{2}(|(S,\bar{S})|+1) \quad (S \subseteq V,\ S \text{ odd}) \quad (5)$$

are valid inequalities for (WPP1).

**Proof.** First observe that the total flow in both directions between S and $\bar{S}$ must be even and at least equal to $|(S,\bar{S})|$, i.e.

$$\sum_{(i,j) \in (S,\bar{S})} (x_{ij} + x_{ji}) \geq |(S,\bar{S})| + 1$$

whenever S is odd. The conclusion follows immediately from the fact that

$$\sum_{(i,j)\in(S,\bar{S})} x_{ij} = \sum_{(i,j)\in(S,\bar{S})} x_{ji}. \qquad \blacksquare$$

It is unnecessary to impose (5) for $\bar{S}$ once it has been imposed for S. It is also interesting to point out that these constraints are analogous to the blossom inequalities for the Chinese Postman Problem [1]. They do not, however, provide us with a complete characterization of the convex hull of feasible solutions to (WPP1). To illustrate, consider the example depicted in Figure 1.

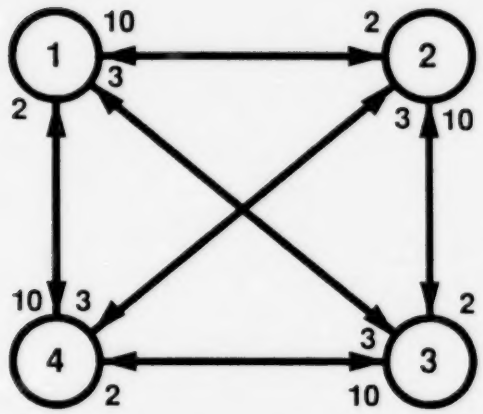


Figure 1. Example for which the solution of the LP relaxation is not integer. The cost of traversing an edge in a given direction is shown on the arrow.

Indeed, solving the problem with constraints (1), (2), (3) and (5) yields the following non-integer solution: $x_{1,2} = x_{2,3} = x_{3,4} = x_{4,1} = 3/2$ and $x_{1,3} = x_{3,1} = x_{2,4} = x_{4,2} = 1/2$.

We now propose an alternative formulation which exploits the fact that a WPT solution may be viewed as a set of elementary dicycles, i.e. directed cycles passing through each vertex at most once. Let H be the set of all dicycles on $D_G$ and $H_{ij} \subset H$, the set of dicycles containing edge ij in either direction. Also define $C_h$, the cost of dicycle h and $v_h$, the flow on h. The formulation is then

(WPP2) $$\text{Minimize} \sum_{h \in H} C_h v_h$$

subject to

$$\sum_{h \in H_{ij}} v_h \geq 1 \qquad (ij \in E) \qquad (6)$$

$$v_h \geq 0 \qquad (h \in H) \qquad (7)$$

$$v_h \text{ integer} \qquad (h \in H). \qquad (8)$$

In this formulation, constraints (6), (7) and (8) are the counterparts of (1), (3) and (4) of (WPP1), respectively. Constraints (2) of (WPP1) are implicitly satisfied by the definition of the cycle variables. Note, that any feasible solution $v = (v_h)$ to (WPP2) translates uniquely into a solution $x = (x_{ij}, x_{ji})$ to (WPP1) and that constraints (5) can be rewritten in terms of the $v_h$ variables. This translation does not, however, mean that (WPP1) and (WPP2) are equivalent. Indeed, constraints (6) are not an exact translation of constraints (1) in terms of cycle variables. In order to see this, first define $h_{ij}$ as the dicycle on $\{i,j\}$ and $H'_{ij} = H_{ij} \setminus \{h_{ij}\}$. Then the following identity holds for any feasible solution:

$$x_{ij} + x_{ji} = \sum_{h \in H'_{ij}} v_h + 2 v_{h_{ij}} = \sum_{h \in H_{ij}} v_h + v_{h_{ij}}. \qquad (9)$$

Constraints (1) state that the left-hand side of (9) is at least equal to 1, whereas constraints (6) apply to the first term of the right-hand side only. We can also state

**Proposition 5.** In any optimal solution, all variables $v_h$ are bounded above by 1.

**Proof.** This result is a direct consequence of the fact that $D_G$ contains no negative cycles and thus, it is never profitable to have $v_h > 1$. ∎

## 3. Algorithmic Approaches

The above formulations naturally lend themselves to well-known exact algorithms. (WPP1) could be solved by a branch-and-cut algorithm (see, for example, Padberg and Rinaldi [4]) in which constraints (5) are generated when they are found to be violated. Moreover, Proposition 3 suggests the use of a three-way branching scheme for fractional variables. As for (WPP2), a set covering approach with column generation is an obvious choice. This approach would appear particularly appropriate in contexts where the presence of side constraints would help reduce the number of potential variables $v_h$. Such constraints cannot be incorporated so easily in (WPP1).

In addition, a number of heuristic approaches are also possible. The first algorithm was proposed by Win [5]. It consists of solving a minimum cost flow problem on an auxiliary graph derived from G. The optimal flow can be translated into $x_{ij}$ variables satisfying (1), (2) and (3). Hence, this algorithm can easily be seen as equivalent to solving the linear relaxation of (WPP). Therefore, if G is Eulerian, Win's algorithm

provides an optimal WPP solution. For non-Eulerian graphs, Win proposes the following heuristic. First transform G into a Eulerian graph G', using the maximum matching procedure of Edmonds and Johnson [1], with edge costs $\bar{c}_{ij} = (c_{ij} + c_{ji})/2$ and then apply the minimum cost flow algorithm on G'. Win provides an upper bound on the difference between the cost of the heuristic solution and that of the optimal WPT solution of G. Moreover, the worst-case ratio of the heuristic is equal to 2 and this bound is asymptotically tight [5].

One major drawback of Win's algorithm is that the solution is conditioned by the choice of edges made when transforming G into G'. This can be avoided by operating directly on G. We therefore propose the following heuristic. Here an arc-pair is defined as an edge ij for which $x_{ij} = x_{ji} = 1$.

Step 1. Apply Win's minimum cost flow algorithm to G. If the solution can be interpreted as a vector $x = (x_{ij}, x_{ji})$ feasible for the WPP, it is then optimal; stop.

Step 2. Round up to 1 all fractional $x_{ij}$ and $x_{ji}$ variables. This solution is feasible for the WPP.

Repeat Step 3 until the list of arc-pairs has been completely examined and then stop.

Step 3. Consider the next arc-pair ij in the list and compute $c'_{ij}$ ($c'_{ji}$) the cost of a shortest dipath from i to j (from j to i). If $c'_{ij} < c_{ij}$ and $c_{ij} - c'_{ji} > c_{ji} - c'_{ji}$, replace (i,j) by the shortest dipath yielding $c'_{ij}$. If this creates new arc-pairs, insert them at the end of the list. ∎

We now derive two results related to this procedure.

**Proposition 6.** The worst-case ratio of the heuristic is equal to 2.

**Proof**. Let $z^*$ be the optimum, $\underline{z}$ the solution value obtained at the end of Step 1, $\tilde{z}$ the solution value obtained at the end of Step 2 and $\bar{z}$ the heuristic solution value. First observe that $\underline{z} \leq z^* \leq \bar{z} \leq \tilde{z}$. Since $D_G$ contains no negative dicycles and fractional variables at the end of Step 1 are always equal to ½, it follows that

(i) $x_{ji} = ½$ whenever $x_{ij} = ½$ because of constraints (1);

(ii) $0 \leq ½ \sum_{ij \in F} (c_{ij} + c_{ji}) \leq \underline{z} \leq z^* \leq \underline{z} + ½ \sum_{ij \in F} (c_{ij} + c_{ji}) = \tilde{z}$, where

$F = \{ij \in E : x_{ij} = x_{ji} = ½ \text{ at the end of Step 1}\}$.

The conclusion follows since the rounding process at most doubles the solution value. ∎

**Corollary 7.** The difference between the heuristic solution value and the optimum is bounded above by $½ \sum_{ij \in F} (c_{ij} + c_{ji})$. ∎

We now show that our procedure is sometimes preferable to Win's algorithm. Our example is precisely that used by Win to illustrate the asymptotic tightness of the worst-case ratio of his heuristic. Consider the graph represented in Figure 2, with the following costs:

$$c_{1,2} = \varepsilon, \; c_{2,1} = M,$$

$$c_{2,3} = c_{3,2} = 1,$$

$$c_{1,i} = M, \; c_{i,1} = \varepsilon \quad \text{for} \quad i = 3, \ldots, 2n+1,$$

$$c_{i,i+1} = 3\varepsilon, \; c_{i+1,i} = 2M \quad \text{for } i = 3, \ldots, 2n,$$

where $\varepsilon$ is an arbitrarily small positive number and M is arbitrarily large.

The solution provided by Win's algorithm has a cost $\bar{z}$ equal to $(4n-3) + 2(2n-1)(3n-1)\varepsilon - (6n-4)\varepsilon$. Solving the minimum cost flow algorithm on the original graph, as suggested, yields an optimal solution of cost $z^* = (2n-1) + (2n-1)(3n-1)\varepsilon$, i.e. the algorithm terminates after Step 1.

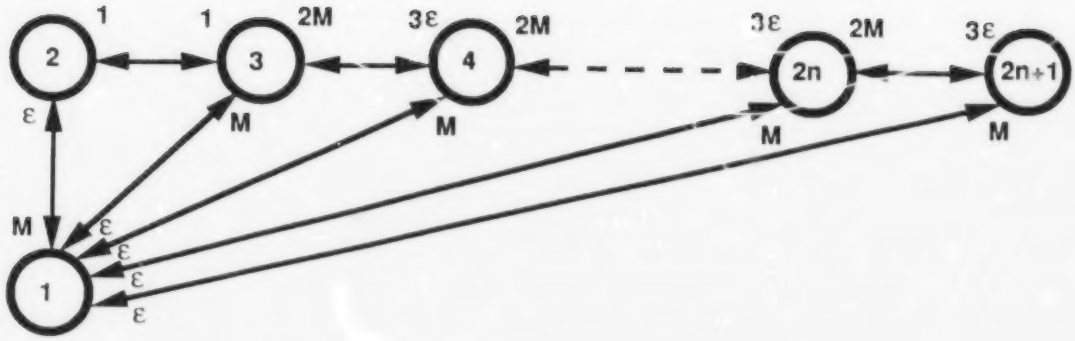


Figure 2. Example for which the proposed procedure is superior to Win's heuristic

## 4. Conclusion

We have considered in this paper the WPP on general graphs. The main contributions of this study are the development of a family of valid cuts for this problem, an alternative ILP formulation and a new heuristic which outperforms in some cases that proposed by Win.

## 5. References

[1] J. Edmonds and E.L. Johnson, "Matching, Euler tours and the Chinese postman problem", *Mathematical Programming* **5**, 88-124 (1973).

[2] M. Guan, "On the windy postman problem", *Discrete Applied Mathematics* **9**, 41-46 (1984).

[3] E. Minieka, "The Chinese postman problem for mixed networks", *Management Science* **25**, 643-648 (1979).

[4] M. Padberg and G. Rinaldi, "Optimization of a 532-city symmetric traveling salesman problem by branch and cut", *Operations Research Letters* **6**, 1-7 (1987).

[5] Z. Win, "On the windy postman problem on Eulerian graphs", *Mathematical Programming* **44**, 97-112 (1989).